

# GLAZMAN-KREIN-NAIMARK THEORY, LEFT-DEFINITE THEORY AND THE SQUARE OF THE LEGENDRE POLYNOMIALS DIFFERENTIAL OPERATOR

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*We dedicate this paper to the memory of W. N. (Norrie) Everitt (1924-2011).*

**ABSTRACT.** As an application of a general left-definite spectral theory, Everitt, Littlejohn and Wellman, in 2002, developed the left-definite theory associated with the classical Legendre self-adjoint second-order differential operator  $A$  in  $L^2(-1, 1)$  which has the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  as eigenfunctions. As a consequence, they explicitly determined the domain  $\mathcal{D}(A^2)$  of the self-adjoint operator  $A^2$ . However, this domain, in their characterization, does not contain boundary conditions. In fact, this is a general feature of the left-definite approach developed by Littlejohn and Wellman. Yet, the square of the second-order Legendre expression is in the limit-4 case at each end point  $x = \pm 1$  in  $L^2(-1, 1)$  so  $\mathcal{D}(A^2)$  should exhibit four boundary conditions. In this paper, we show that this domain can, in fact, be expressed using four separated boundary conditions using the classical GKN (Glazman-Krein-Naimark) theory. In addition, we determine a new characterization of  $\mathcal{D}(A^2)$  that involves four *non-GKN* boundary conditions. These new boundary conditions are surprisingly simple - and natural - and are equivalent to the boundary conditions obtained from the GKN theory.

## 1. INTRODUCTION

The analytical study of the classical second-order Legendre differential expression

$$\ell[y](x) = -((1 - x^2)y'(x))'$$

has a long and rich history stretching back to the seminal work of H. Weyl in 1910 [23] and E. C. Titchmarsh in 1940 [22]. Part, if not most, of the reason for the importance of this second-order expression lies in the fact that the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  are solutions. More specifically, the Legendre polynomial  $y = P_n(x)$ , for  $n \in \mathbb{N}_0$ , is a solution of the eigenvalue equation

$$\ell[y](x) = n(n + 1)y(x).$$

In the Hilbert space  $L^2(-1, 1)$ , there is a continuum of self-adjoint operators generated by  $\ell[\cdot]$ . One such operator  $A$  stands out from the rest: this is the Legendre polynomials operator, so named because the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  are eigenfunctions of  $A$ . We review properties of this operator in Section 2.

In the mid 1970's, Å. Pleijel wrote two papers (see [18] and [19]) on the Legendre expression from a left-definite spectral point of view. W. N. Everitt's contribution [8] continued this left-definite study in addition to detailing an in-depth analysis of the Legendre expression in the right-definite setting  $L^2(-1, 1)$  where he discovered new properties of functions in the domain  $\mathcal{D}(A)$  of  $A$ . In [14], A. M. Krall and Littlejohn considered properties of the Legendre expression under the left-definite energy norm. In 2000, R. Vonhoff extended Everitt's results in [20] with an extensive study of  $\ell[\cdot]$  in its (first) left-definite setting. In 2002, Everitt, Littlejohn and Marić [10] published further results in which they gave several equivalent conditions for functions to belong to  $\mathcal{D}(A)$ ; this result is given below in Theorem 1. We also refer the reader to the

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paper [16] by Littlejohn and Zettl where the authors determine all self-adjoint operators, generated by the Legendre expression  $\ell[\cdot]$ , in the Hilbert spaces  $L^2(-1, 1)$ ,  $L^2(-\infty, -1)$ ,  $L^2(1, \infty)$  and  $L^2(\mathbb{R})$ .

Littlejohn and Wellman [15], in 2002, developed a general left-definite theory for an unbounded self-adjoint operator  $T$  bounded below by a positive constant in a Hilbert space  $H = (V, (\cdot, \cdot))$ , where  $V$  denotes the underlying (algebraic) vector space and  $H$  is the resulting topological space induced by the norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$ . In a nutshell, the authors construct a continuum of Hilbert spaces  $\{H_r = (V_r, (\cdot, \cdot)_r)\}_{r>0}$ , forming a Hilbert scale, generated by positive powers of  $T$ . The authors called these Hilbert spaces *left-definite spaces*; they are constructed using the Hilbert space spectral theorem (see [21]) for self-adjoint operators.

It is a difficult problem, in general, to explicitly determine the domain of a power of an unbounded operator. However, the authors in [15] prove that  $V_r = \mathcal{D}(T^{r/2})$  and  $(f, g)_r = (T^{r/2}f, T^{r/2}g)$ . Furthermore, in many practical applications, as the authors demonstrate in [15], the computation of the vector spaces  $V_r$  and inner products  $(\cdot, \cdot)_r$  is surprisingly not difficult. In a subsequent paper, Everitt, Littlejohn and Wellman [11] applied this theory to the Legendre polynomials operator  $A$ . Among other results, the authors explicitly compute the domains of  $\mathcal{D}(A^{n/2})$  for each  $n \in \mathbb{N}$ . Specifically, they proved

$$(1.1) \quad \mathcal{D}(A^{n/2}) = \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', \dots, f^{(n-1)} \in AC_{\text{loc}}(-1, 1); (1-x^2)^{n/2}f^{(n)} \in L^2(-1, 1)\} \quad (n \in \mathbb{N}).$$

In particular, we see that  $\mathcal{D}(A^2)$  is explicitly given by

$$(1.2) \quad B = \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); (1-x^2)^2f^{(4)} \in L^2(-1, 1)\};$$

the reason for using the notation  $B$ , instead of  $\mathcal{D}(A^2)$ , will be made clear shortly. Of course, for  $f \in B$ , we have  $A^2f = \ell^2[f]$ , where  $\ell^2[\cdot]$  is the square of the Legendre differential expression given by

$$(1.3) \quad \ell^2[y](x) = ((1-x^2)^2y''(x))' - 2((1-x^2)y'(x))'.$$

Notice that, curiously, there are no ‘boundary conditions’ given in (1.2). From the Glazman-Krein-Naimark (GKN) theory [17, Theorem 4, Section 18.1], there should be *four* such boundary conditions. This begs an obvious question: how can we ‘extract’ boundary conditions from the representation of  $\mathcal{D}(A^2)$  in (1.2)? In this paper, we will answer this question. It is interesting that the condition  $(1-x^2)^2f^{(4)} \in L^2(-1, 1)$  seems to ‘encode’ these boundary conditions. In fact, along the way, we will characterize  $\mathcal{D}(A^2)$  in four different ways. Of course, we have the algebraic definition

$$(1.4) \quad \mathcal{D}(A^2) := \{f \in \mathcal{D}(A) \mid Af \in \mathcal{D}(A)\}$$

(we will show that  $\mathcal{D}(A^2)$ , given in (1.4), is equal to  $B$ , defined in (1.2)). We will also prove that  $\mathcal{D}(A^2)$  is characterized by GKN boundary conditions associated with a self-adjoint operator  $S$ , generated by  $\ell^2[\cdot]$ , in  $L^2(-1, 1)$ . Specifically, we prove that  $\mathcal{D}(A^2)$  is equal to

$$(1.5) \quad \mathcal{D}(S) := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); f, \ell^2[f] \in L^2(-1, 1); \\ \lim_{x \rightarrow \pm 1} [f, 1]_2(x) = 0; \lim_{x \rightarrow \pm 1} [f, x]_2(x) = 0\},$$

where  $[\cdot, \cdot]_2$  is the sesquilinear form associated with Green’s formula and  $\ell^2[\cdot]$  in  $L^2(-1, 1)$ ; this form will be defined in Section 4. In this paper, we also show that  $\mathcal{D}(A^2)$  is equal to

$$(1.6) \quad D := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); f, \ell^2[f] \in L^2(-1, 1); \\ \lim_{x \rightarrow \pm 1} (1-x^2)f'(x) = 0; \lim_{x \rightarrow \pm 1} ((1-x^2)^2f''(x))' = 0\}.$$

This characterization of  $\mathcal{D}(A^2)$  is surprising since the boundary conditions in (1.6) are *not* GKN boundary conditions; we say that  $D$  is a GKN-like domain. The boundary conditions in (1.6) are remarkably simple; indeed, they are obtained as limits from each of the two terms in (1.3) minus one derivative.

In [9], the authors first showed the smoothness condition

$$(1.7) \quad f \in \mathcal{D}(A) \Rightarrow f' \in L^2(-1, 1).$$

As a consequence of our results in this paper, we are able to generalize (1.7) by proving

$$f \in \mathcal{D}(A^2) \Rightarrow f'' \in L^2(-1, 1) \text{ and } \ell[f] \in AC[-1, 1];$$

see Corollary 1 below.

The contents of this paper are as follows. In Section 2, we discuss properties of the Legendre expression and the Legendre polynomials operator  $A$  in  $L^2(-1, 1)$ . Section 3 deals briefly with the *algebraic* definition of the square  $A^2$  of  $A$ . In Section 4, we define a self-adjoint operator  $S$  using the GKN Theory; this operator  $S$  will ultimately be shown to be  $A^2$ . The main theorems proven in this paper are stated in Section 5. A key and indispensable analytic tool - the Chisholm-Everitt Theorem - used in the proofs of these theorems is discussed in Section 6. The proof that  $\mathcal{D}(A^2) = \mathcal{D}(S)$  is given in Section 7. Section 8 establishes the proof that  $B = \mathcal{D}(S)$ . In Section 9, we show that  $\mathcal{D}(S) = D$ . The proofs of the theorems in these last three sections establish our main result, Theorem 6, which we state in Section 5. Lastly, in Section 10, we conjecture a generalization of our main results. Further details on all of the results contained in this manuscript can be found in the Ph.D. thesis [24] of Quinn Wicks.

One final remark: to summarize, in this paper we show that our left-definite characterization (1.2) of  $\mathcal{D}(A^2)$  can be rewritten as a GKN domain (Theorem 4) and as a GKN-like domain (Theorem 5). Presumably, techniques developed in this paper will establish, for  $n \in \mathbb{N}$ , that the left-definite characterization  $\mathcal{D}(A^n)$ , given in (1.1), can be expressed as both a GKN domain and a GKN-like domain. However, it is important to note - see (1.1) - that the left-definite theory also explicitly determines the domains  $\mathcal{D}(A^{n/2})$  of  $A^{n/2}$  for odd, positive integers  $n$ . The GKN theory was not built to handle these operators or domains.

## 2. THE LEGENDRE DIFFERENTIAL EXPRESSION AND THE LEGENDRE POLYNOMIALS SELF-ADJOINT OPERATOR $A$

The classic second-order Legendre differential expression is defined by

$$(2.1) \quad \ell[y](x) := -((1-x^2)y'(x))' \quad (\text{a.e. } x \in (-1, 1)).$$

The maximal operator, associated with  $\ell[\cdot]$  in  $L^2(-1, 1)$ , is defined by

$$\begin{aligned} T_{1,\max}f &= \ell[f] \\ f &\in \Delta_{1,\max}, \end{aligned}$$

where  $\Delta_{1,\max}$  is the maximal domain, defined by

$$(2.2) \quad \Delta_{1,\max} := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1, 1); f, \ell[f] \in L^2(-1, 1)\}.$$

The corresponding minimal operator  $T_{1,\min}$  is defined to be

$$\begin{aligned} T_{1,\min}f &= \ell[f] \\ f &\in \mathcal{D}(T_{1,\min}), \end{aligned}$$

where  $\mathcal{D}(T_{1,\min})$  is the minimal domain given by

$$\mathcal{D}(T_{1,\min}) := \{f \in \Delta_{1,\max} \mid [f, g]_1(x)|_\alpha^\beta = 0 \text{ for all } g \in \Delta_{1,\max}\}.$$

We note that this operator  $T_{1,\min}$  is a closed, symmetric operator. Furthermore,  $T_{1,\max}$  and  $T_{1,\min}$  are adjoints of each other.

Green's formula, for an arbitrary compact subinterval  $[\alpha, \beta]$  of  $(-1, 1)$  and  $f, g \in \Delta_{1,\max}$ , is given by

$$\int_\alpha^\beta \ell[f](x)\overline{g}(x)dx - \int_\alpha^\beta f(x)\overline{\ell[g]}(x)dx = [f, g]_1(x)|_\alpha^\beta,$$

where the sesquilinear form  $[\cdot, \cdot]_1$  is defined by

$$(2.3) \quad [f, g]_1(x) := -(1 - x^2)(f'(x)\overline{g}(x) - f(x)\overline{g}'(x)) \quad (f, g \in \Delta_{1, \max}).$$

By definition of  $\Delta_{1, \max}$  and Hölder's inequality, we see that the limits

$$\lim_{x \rightarrow \pm 1} [f, g](x)$$

exist and are finite for all  $f, g \in \Delta_{1, \max}$ .

The endpoints  $x = \pm 1$  are both regular singular endpoints, in the sense of Frobenius, of  $\ell[\cdot]$  and it is well-known that this expression is in the limit-circle case at each endpoint. Consequently, the deficiency index of the minimal operator  $T_{1, \min}$  is  $(2, 2)$ . This implies that there is a continuum of self-adjoint restrictions of  $T_{1, \max}$ . The GKN Theorem [17, Theorem 4, Section 18.1] (see also [1, Volume II, Chapter 8] and [6, Chapter XIII]) provides a ‘recipe’ for determining each of these operators. We are interested in that particular self-adjoint restriction  $A$  which has the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  as eigenfunctions.

This Legendre polynomials operator  $A : \mathcal{D}(A) \subset L^2(-1, 1) \rightarrow L^2(-1, 1)$  is specifically given by

$$(2.4) \quad \begin{aligned} Af &= \ell[f] \\ f &\in \mathcal{D}(A), \end{aligned}$$

where

$$(2.5) \quad \mathcal{D}(A) := \{f \in \Delta_{1, \max} \mid \lim_{x \rightarrow \pm 1} (1 - x^2)f'(x) = 0\}.$$

We note that the boundary conditions expressed in (2.5) are equivalent to

$$[f, 1]_1(\pm 1) = 0 \quad (f \in \Delta_{1, \max}).$$

Furthermore, it is well known that the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  form a complete (orthogonal) set of eigenfunctions of  $A$  and the spectrum  $\sigma(A)$  is discrete and given explicitly by

$$\sigma(A) := \{n(n+1) \mid n \in \mathbb{N}_0\}.$$

For our purposes, it is the case that

$$(Af, f) = \int_{-1}^1 \ell[f](x)\overline{f}(x)dx = \int_{-1}^1 (1 - x^2)|f'(x)|^2 dx \geq 0 \quad (f \in \mathcal{D}(A));$$

that is to say,  $A$  is a positive operator. The positivity of  $A$  implies that the left-definite theory developed by Littlejohn and Wellman in [15] can be used to determine  $\mathcal{D}(A^n)$  for each  $n \in \mathbb{N}$ ; indeed, see (1.1).

The following theorem, shown by Everitt, Littlejohn and Marić in [10], lists several equivalent conditions for a function  $f$  to belong to  $\mathcal{D}(A)$ . Note the surprising, and remarkable, equivalence of conditions (ii) and (iii) (and (ii) and (v)) below; parts (ii) and (v) will be of particular use to us in this paper.

**Theorem 1.** *Let  $f \in \Delta_{1, \max}$ , where  $\Delta_{1, \max}$  is given in (2.2). The following conditions are equivalent:*

- (i)  $f \in \mathcal{D}(A)$ ;
- (ii)  $f' \in L^2(-1, 1)$ ;
- (iii)  $f' \in L^1(-1, 1)$ ;
- (iv)  $f$  is bounded on  $(-1, 1)$ ;
- (v)  $f \in AC[-1, 1]$ ;
- (vi)  $(1 - x^2)^{1/2}f' \in L^2(-1, 1)$ ;
- (vii)  $(1 - x^2)f'' \in L^2(-1, 1)$ .

## 3. THE SQUARE OF THE LEGENDRE POLYNOMIALS OPERATOR

The square  $A^2 : \mathcal{D}(A^2) \subset L^2(-1, 1) \rightarrow L^2(-1, 1)$  of the Legendre polynomials operator  $A$  in  $L^2(-1, 1)$  is *algebraically* defined by

$$(3.1) \quad A^2 f := \ell^2[f]$$

for  $f \in \mathcal{D}(A^2)$ , where  $\mathcal{D}(A^2)$  is defined in (1.4), and where

$$(3.2) \quad \begin{aligned} \ell^2[y](x) &:= ((1-x^2)^2 y''(x))'' - 2((1-x^2)y'(x))' \\ &= (1-x^2)^2 y^{(4)}(x) - 8x(1-x^2)y'''(x) + (14x^2-6)y''(x) + 4xy'(x). \end{aligned}$$

By standard results from functional analysis (specifically, the Hilbert space spectral theorem), it can be shown that  $A^2$  is a self-adjoint operator in  $L^2(-1, 1)$ , the spectrum of  $A^2$  is given by  $\sigma(A^2) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}$  and the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  are eigenfunctions of  $A^2$ .

It is natural to ask whether we can explicitly describe the functions in the domain  $\mathcal{D}(A^2)$  similar to how we characterize elements in  $\mathcal{D}(A)$  as in (2.5) (or by Theorem 1). In the next section, we identify  $A^2$  with a self-adjoint operator  $S$  obtained through an application of the GKN theory.

## 4. A GKN SELF-ADJOINT OPERATOR GENERATED BY THE SQUARE OF THE LEGENDRE DIFFERENTIAL EXPRESSION

The maximal domain  $\Delta_{2,\max}$  in  $L^2(-1, 1)$  associated with the square of the Legendre expression  $\ell^2[\cdot]$ , defined in (3.2), is given by

$$(4.1) \quad \Delta_{2,\max} := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); f, \ell^2[f] \in L^2(-1, 1)\}.$$

The sesquilinear form  $[\cdot, \cdot]_2(\cdot) : \Delta_{2,\max} \times \Delta_{2,\max} \times (-1, 1)$ , associated with  $\ell^2[\cdot]$ , is defined by

$$(4.2) \quad \begin{aligned} [f, g]_2(x) &:= ((1-x^2)^2 f''(x))' \overline{g}(x) - ((1-x^2)^2 \overline{g}''(x))' f(x) \\ &\quad - (1-x^2)^2 f''(x) \overline{g}'(x) + (1-x^2)^2 f'(x) \overline{g}''(x) \\ &\quad - 2(1-x^2) f'(x) \overline{g}(x) + 2(1-x^2) f(x) \overline{g}'(x) \quad (x \in (-1, 1)). \end{aligned}$$

For  $f, g \in \Delta_{2,\max}$  and  $[\alpha, \beta] \subset (-1, 1)$ , Green's formula for  $\ell^2[\cdot]$  is given by

$$(4.3) \quad \int_\alpha^\beta \ell^2[f](x) \overline{g}(x) dx - \int_\alpha^\beta f(x) \overline{\ell^2[g]}(x) dx = [f, g]_2(x) \Big|_\alpha^\beta.$$

By definition of  $\Delta_{2,\max}$  and Hölder's inequality, we see that the limits

$$[f, g]_2(\pm 1) := \lim_{x \rightarrow \pm 1} [f, g]_2(x)$$

exist and are finite for all  $f, g \in \Delta_{2,\max}$ . Clearly

$$(4.4) \quad P_n \in \Delta_{2,\max} \quad (n \in \mathbb{N}_0),$$

where  $P_n(x)$  is the  $n^{\text{th}}$  degree Legendre polynomial. In particular, the functions 1 and  $x$  belong to  $\Delta_{2,\max}$ .

The endpoints  $x = \pm 1$  are both regular singular points, in the sense of Frobenius, of  $\ell^2[\cdot]$ . The Frobenius indicial equation, at either endpoint, is given by

$$r^2(r-1)^2 = 0.$$

It follows, from the general Weyl theory, that each endpoint is in the limit-4 case so the deficiency index of the minimal operator  $T_{2,\min}$ , generated by  $\ell^2[\cdot]$ , in  $L^2(-1, 1)$  is  $(4, 4)$ . Consequently, each self-adjoint operator, generated by  $\ell^2[\cdot]$ , in  $L^2(-1, 1)$  is determined by restricting  $\Delta_{2,\max}$  to four boundary conditions of the form

$$(4.5) \quad [f, f_j]_2(1) - [f, f_j]_2(-1) = 0,$$

where  $[\cdot, \cdot]_2$  is given in (4.2) and where  $\{f_1, f_2, f_3, f_4\} \subset \Delta_{2,\max}$  is linearly independent modulo the minimal domain  $\Delta_{2,\min}$  defined by

$$\Delta_{2,\min} := \{f \in \Delta_{2,\max} \mid [f, g]_2|_{-1}^1 = 0 \text{ for all } g \in \Delta_{2,\max}\}.$$

We now identify a particular self-adjoint operator restriction  $S$  of  $T_{\max}$ , generated by  $\ell^2[\cdot]$ , having the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  as a complete set of eigenfunctions.

For  $j = 1, 2, 3, 4$ , define  $f_j \in \Delta_{2,\max} \cap C^4[-1, 1]$  by

$$(4.6) \quad \begin{aligned} f_1(x) &= \begin{cases} 1 & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} & f_2(x) &= \begin{cases} 0 & \text{near } x = 1 \\ 1 & \text{near } x = -1, \end{cases} \\ f_3(x) &= \begin{cases} x & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} & f_4(x) &= \begin{cases} 0 & \text{near } x = 1 \\ x & \text{near } x = -1, \end{cases} \end{aligned}$$

**Proposition 1.** *The functions  $\{f_j\}_{j=1}^4$ , defined in (4.6), are linearly independent modulo  $\Delta_{2,\min}$ .*

*Proof.* Calculations show that the functions  $\ln(1 \pm x)$  and  $(1 \pm x)\ln(1 \pm x)$  belong to  $\Delta_{2,\max}$ . We modify these functions by defining the four functions  $g_j \in \Delta_{2,\max} \cap C^4(-1, 1)$  ( $j = 1, 2, 3, 4$ )

$$\begin{aligned} g_1(x) &= \begin{cases} \ln(1-x) & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} & g_2(x) &= \begin{cases} 0 & \text{near } x = 1 \\ \ln(1+x) & \text{near } x = -1, \end{cases} \\ g_3(x) &= \begin{cases} (1-x)\ln(1-x) & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} & g_4(x) &= \begin{cases} 0 & \text{near } x = 1 \\ (1+x)\ln(1+x) & \text{near } x = -1. \end{cases} \end{aligned}$$

Suppose that

$$\sum_{j=1}^4 \alpha_j f_j \in \Delta_{2,\min};$$

then, by definition of  $\Delta_{2,\min}$ , we see that

$$(4.7) \quad \left[ \sum_{j=1}^4 \alpha_j f_j, g \right]_2 \Big|_{-1}^1 = 0 \quad (g \in \Delta_{2,\max}),$$

where  $[\cdot, \cdot]_2$  is the sesquilinear form defined in (4.2). A calculation shows that

$$0 = \left[ \sum_{j=1}^4 \alpha_j f_j, g_1 \right]_2 \Big|_{-1}^1 = -4\alpha_3$$

so  $\alpha_3 = 0$ . Similarly, we find that  $\alpha_1 = \alpha_2 = \alpha_4 = 0$  after substituting  $g = g_2, g_3, g_4$  into (4.7). This completes the proof.  $\square$

It is clear that the boundary conditions

$$[f, f_1]_2(1) = [f, f_3]_2(1) = [f, f_2]_2(-1) = [f, f_4]_2(-1) = 0$$

are equivalent to the boundary conditions

$$[f, 1]_2(\pm 1) = [f, x]_2(\pm 1) = 0.$$

We are now in position to define the operator  $S$  which we show later (see Section 7) to be equal to the operator  $A^2$ , given in (3.1) and (1.4). Indeed, let  $S : \mathcal{D}(S) \subset L^2(-1, 1) \rightarrow L^2(-1, 1)$  be defined by

$$(4.8) \quad \begin{aligned} Sf &= \ell^2[f] := \ell[\ell[f]] \\ f &\in \mathcal{D}(S), \end{aligned}$$

where the domain  $\mathcal{D}(S)$  of  $S$  is defined in (1.5). By the GKN Theorem [17, Theorem 4, Section 18.1],  $S$  is self-adjoint in  $L^2(-1, 1)$ . Moreover, notice that for  $f \in \Delta_{2, \max}$ ,

$$(4.9) \quad [f, 1]_2(x) = ((1 - x^2)^2 f''(x))' - 2(1 - x^2) f'(x)$$

and

$$(4.10) \quad \begin{aligned} [f, x]_2(x) &= ((1 - x^2)^2 f''(x))' x - (1 - x^2)^2 f''(x) - 2x(1 - x^2) f'(x) + 2(1 - x^2) f(x) \\ &= x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x). \end{aligned}$$

From (4.9) and (4.10), it is easy to see that the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  satisfy

$$[P_n, 1]_2(\pm 1) = [P_n, x]_2(\pm 1) = 0.$$

That is to say, the Legendre polynomials  $\{P_n\}_{n=0}^\infty \subset \mathcal{D}(S)$ . Moreover  $\ell^2[P_n] = \ell[\ell[P_n]] = n(n+1)\ell[P_n] = n^2(n+1)^2 P_n$  ( $n \in \mathbb{N}_0$ ).

From [17] and standard results in spectral theory, the following result holds.

**Theorem 2.** *The operator  $S$ , defined in (4.8) and (1.5), is an unbounded self-adjoint operator in  $L^2(-1, 1)$ . The Legendre polynomials  $\{P_n\}_{n=0}^\infty$  form a complete set of (orthogonal) eigenfunctions of  $S$  in  $L^2(-1, 1)$ . The spectrum  $\sigma(S)$  of  $S$  is discrete and given explicitly by*

$$\sigma(S) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}.$$

## 5. STATEMENTS OF THE MAIN THEOREMS

There are four main theorems that we prove in this paper.

**Theorem 3.** *Let  $\mathcal{D}(A^2)$  and  $\mathcal{D}(S)$  be given, respectively, as in (1.4) and (1.5). Then*

$$\mathcal{D}(A^2) = \mathcal{D}(S).$$

*Proof.* see Section 7. □

**Theorem 4.** *Let  $B$  and  $\mathcal{D}(S)$  be given, respectively, as in (1.2) and (1.5). Then*

$$B = \mathcal{D}(S).$$

*Proof.* see Section 8. □

**Theorem 5.** *Let  $\mathcal{D}(S)$  and  $D$  be given, respectively, as in (1.5) and (1.6). Then*

$$D = \mathcal{D}(S).$$

*Proof.* see Section 9. □

From these three theorems, we obtain our main result, namely

**Theorem 6.** *Let  $\Delta_{2, \max}$ , given in (4.1), be the maximal domain of the formal square  $\ell^2[\cdot]$  of the Legendre differential expression defined by*

$$\ell^2[y](x) = ((1 - x^2)^2 y''(x))'' - 2((1 - x^2) y'(x))' \quad (x \in (-1, 1))$$

*and let  $[\cdot, \cdot]_2$  be the associated sesquilinear form for  $\ell^2[\cdot]$  given in (4.2). Define the operator  $T : \mathcal{D}(T) \subset L^2(-1, 1) \rightarrow L^2(-1, 1)$  by*

$$(Tf)(x) = \ell^2[f](x) \quad (\text{a.e. } x \in (-1, 1))$$

$$f \in \mathcal{D}(T) := \mathcal{D}(A^2),$$

*where  $\mathcal{D}(A^2)$ , algebraically defined in (1.4), is the domain of the square of the Legendre polynomials operator  $A$  defined in (2.5). That is to say,  $T$  is the square of the classical Legendre polynomials operator  $A$ , given in (2.4) and (2.5). Then the following statements are equivalent:*

- (i)  $f \in \mathcal{D}(T)$ ;
- (ii)  $f, f', f'', f''' \in AC_{\text{loc}}(-1, 1)$  and  $(1 - x^2)^2 f^{(4)} \in L^2(-1, 1)$ ;
- (iii)  $f \in \Delta_{2, \max}$  and  $[f, 1]_2(\pm 1) = [f, x]_2(\pm 1) = 0$ ;
- (iv)  $f \in \Delta_{2, \max}$  and  $\lim_{x \rightarrow \pm 1} (1 - x^2) f'(x) = \lim_{x \rightarrow \pm 1} ((1 - x^2)^2 f''(x))' = 0$ .

Moreover,  $T$  is a self-adjoint operator in  $L^2(-1, 1)$  having the Legendre polynomials  $\{P_n\}_{n=0}^\infty$  as a complete set of eigenfunctions in  $L^2(-1, 1)$  and having discrete spectrum  $\sigma(T^2)$  explicitly given by

$$\sigma(T^2) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}.$$

## 6. A KEY INTEGRAL INEQUALITY

A key result in our analysis below is the following operator inequality established by Chisholm and Everitt (CE) in [5].

**Theorem 7.** (*The CE Theorem*) Let  $(a, b)$  be an open interval of the real line (bounded or unbounded) and let  $w$  be a Lebesgue measurable function that is positive a.e.  $x \in (a, b)$ . Suppose  $\varphi, \psi : (a, b) \rightarrow \mathbb{C}$  satisfy the conditions

- (i)  $\varphi, \psi \in L^2_{\text{loc}}((a, b); w)$ ;
- (ii) there exists  $c \in (a, b)$  such that  $\varphi \in L^2((a, c]; w)$  and  $\psi \in L^2([c, b); w)$ ;
- (iii) for all  $[\alpha, \beta] \subset (a, b)$

$$\int_\alpha^\beta |\varphi(x)|^2 w(x) dx > 0 \text{ and } \int_\alpha^\beta |\psi(x)|^2 w(x) dx > 0.$$

Define the linear operators  $A, B : L^2((a, b); w) \rightarrow L^2_{\text{loc}}((a, b); w)$  by

$$(Af)(x) = \varphi(x) \int_x^b \psi(t) f(t) w(t) dt \quad (t \in (a, b); f \in L^2((a, b); w)),$$

and

$$(Bf)(x) = \psi(x) \int_a^x \varphi(t) f(t) w(t) dt \quad (t \in (a, b); f \in L^2((a, b); w)).$$

Let  $K : (a, b) \rightarrow (0, \infty)$  be given by

$$(6.1) \quad K(x) := \left( \int_a^x |\varphi(t)|^2 w(t) dt \right)^{1/2} \left( \int_x^b |\psi(t)|^2 w(t) dt \right)^{1/2} \quad (t \in (a, b)),$$

and define  $K \in [0, \infty]$  by

$$(6.2) \quad K := \sup\{K(x) \mid x \in (a, b)\}.$$

Then a necessary and sufficient condition that  $A$  and  $B$  are both bounded operators from  $L^2((a, b); w)$  into  $L^2((a, b); w)$  is that

$$0 < K < \infty.$$

Moreover, the following inequalities hold

$$(6.3) \quad \|Af\| \leq 2K \|f\| \quad (f \in L^2((a, b); w))$$

$$(6.4) \quad \|Bg\| \leq 2K \|g\| \quad (g \in L^2((a, b); w))$$

where the number  $K$  is defined by (6.2). In general, the number  $2K$  appearing in both (6.3) and (6.4) is best possible for these inequalities to hold.



**Remark 1.** *Theorem 7, proven by Chisholm and Everitt in 1970, was extended in 1999 by Chisholm, Everitt and Littlejohn to the spaces  $L^p((a, b); w)$  and  $L^q((a, b); w)$  where  $p, q > 1$  are conjugate indices; see [9]. Both Theorem 7 and its generalization in [9] have seen several applications including a new proof of the classical Hardy integral inequality [13, Section 9.8, Theorem 327] (see [9, Example 1]) and numerous applications to orthogonal polynomials (for example, see [9, Section 6]). Several more applications of the CE Theorem will be given in this paper. Indeed, Theorem 7 proves to be an indispensable tool in our analysis below.*

## 7. PROOF OF THEOREM 3

We now prove Theorem 3, namely that  $\mathcal{D}(A^2) = \mathcal{D}(S)$ , where  $\mathcal{D}(A^2)$  is defined in (1.4) and  $\mathcal{D}(S)$  is given in (1.5). Throughout this section, we assume that  $f$  is a real-valued function on  $(-1, 1)$ .

*Proof.*  $\mathcal{D}(S) \subset \mathcal{D}(A^2)$ :

Let  $f \in \mathcal{D}(S)$ . We know that

- (i)  $f, f', f'', f''' \in AC_{\text{loc}}(-1, 1)$ ;
- (ii)  $f \in L^2(-1, 1)$ ;
- (iii)  $\ell^2[f] \in L^2(-1, 1)$  where  $\ell^2[\cdot]$  is defined by (3.2);
- (iv)  $[f, 1]_2(\pm 1) = 0$ , where  $[\cdot, 1]_2(\cdot)$  is given in (4.9);
- (v)  $[f, x]_2(\pm 1) = 0$ , where  $[\cdot, x]_2(\cdot)$  is given in (4.10).

Taking into account the definition of  $\mathcal{D}(A)$  in (2.5) and  $\mathcal{D}(A^2)$  in (1.4), we need to show that

- (a)  $f, f' \in AC_{\text{loc}}(-1, 1)$ ;
- (b)  $f \in L^2(-1, 1)$ ;
- (c)  $\ell[f] = -((1 - x^2)f')' = -(1 - x^2)f'' + 2xf' \in L^2(-1, 1)$ ; in fact we will show that  $\ell[f] \in AC[-1, 1]$ ;
- (d)  $\lim_{x \rightarrow \pm 1} (1 - x^2)f'(x) = 0$ ;
- (e)  $\ell[f], \ell'[f] \in AC_{\text{loc}}(-1, 1)$ ;
- (f)  $\ell^2[f] \in L^2(-1, 1)$ ;
- (g)  $\lim_{x \rightarrow \pm 1} (1 - x^2)\ell'[f] = \lim_{x \rightarrow \pm 1} (1 - x^2)((1 - x^2)f'''(x) - 4xf''(x) - 2f'(x)) = 0$ .

Clearly, (a), (b) and (f) are satisfied. As for (g), note that

$$\begin{aligned}
 -(1 - x^2)\ell'[f](x) &= (1 - x^2)^2 f'''(x) - 4x(1 - x^2)f''(x) - 2(1 - x^2)f'(x) \\
 (7.1) \quad &= ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \\
 &= [f, 1]_2(x)
 \end{aligned}$$

so (g) follows from (iv) above. Moreover, by (i) and the fact that the product of a polynomial and a function  $g \in AC_{\text{loc}}(-1, 1)$  also belongs to  $AC_{\text{loc}}(-1, 1)$ , we see that (e) follows. To show (c) note that, by (iii),

$$(7.2) \quad \ell^2[f](x) = \ell[\ell[f]](x) = -((1 - x^2)\ell'[f](x))' \in L^2(-1, 1).$$

We now apply the CE Theorem on the interval  $[0, 1)$  with  $\psi(x) = 1$ ,  $\varphi(x) = 1/(1 - x^2)$  and  $w(x) = 1$ ; note that  $\varphi \in L^2(0, 1/2]$  and  $\psi \in L^2[1/2, 1)$ . A calculation shows that

$$K^2(x) = \int_0^x \frac{dt}{(1 - t^2)^2} \cdot \int_x^1 dt \quad (x \in (0, 1))$$

is bounded on  $(0, 1)$ . Hence we see, from Theorem 7, that

$$\varphi(x) \int_x^1 \psi(t) \ell^2[f](t) w(t) dt = \frac{1}{1 - x^2} \int_x^1 \ell^2[f](t) dt \in L^2[0, 1).$$

That is to say, by (7.2),

$$(7.3) \quad \frac{1}{1-x^2} \left( (1-x^2)\ell'[f](x) - \lim_{x \rightarrow 1} (1-x^2)\ell'[f](x) \right) \in L^2[0, 1).$$

By (iv) and (7.1), we know

$$\lim_{x \rightarrow 1} (1-x^2)\ell'[f](x) = 0.$$

Hence, (7.3) simplifies to

$$\ell'[f] \in L^2[0, 1).$$

A similar application of the CE Theorem on  $(-1, 0]$  reveals that  $\ell'[f] \in L^2(-1, 0]$  and thus we see that

$$\ell'[f] \in L^2(-1, 1).$$

It follows that

$$\ell[f] \in AC[-1, 1] \subset L^2(-1, 1),$$

establishing (c). It remains to show that (d) holds. To this end, observe, from (2.1) and (3.2) that

$$\left( (1-x^2)^2 f''(x) \right)'' = \ell^2[f](x) - 2\ell[f](x).$$

Consequently, from (c) and (f),

$$\left( (1-x^2)^2 f''(x) \right)'' \in L^2(-1, 1)$$

from which we see that

$$\left( (1-x^2)^2 f''(x) \right)', (1-x^2)^2 f''(x) \in AC[-1, 1].$$

In particular, we see that the limits

$$(7.4) \quad \lim_{x \rightarrow \pm 1} \left( (1-x^2)^2 f''(x) \right)'$$

and

$$(7.5) \quad \lim_{x \rightarrow \pm 1} (1-x^2)^2 f''(x)$$

exist and are finite. Moreover, from (iv), (v) and (4.10), we see that

$$(7.6) \quad 0 = \lim_{x \rightarrow \pm 1} (x[f, 1]_2(x) - [f, x]_2(x)) = \lim_{x \rightarrow \pm 1} \left( (1-x^2)^2 f''(x) - 2(1-x^2)f(x) \right).$$

In concert with (7.5), we can say that

$$\lim_{x \rightarrow \pm 1} (1-x^2)f(x) := r$$

exists and is finite. We claim that  $r = 0$ ; to show this, we deal with the limit as  $x \rightarrow 1$ ; a similar proof can be made as  $x \rightarrow -1$ . Suppose, to the contrary, that  $r \neq 0$ ; without loss of generality, suppose  $r > 0$ . Then there exists  $x^* > 0$  such that

$$(1-x^2)f(x) \geq \frac{r}{2} \text{ for } x \in [x^*, 1).$$

However, in this case, we see that

$$\infty > \int_{-1}^1 |f(x)|^2 dx \geq \int_{x^*}^1 |f(x)|^2 dx \geq \left( \frac{r}{2} \right)^2 \int_{x^*}^1 \frac{dx}{(1-x^2)^2} = \infty,$$

contradicting (ii). Hence it follows that

$$\lim_{x \rightarrow \pm 1} (1-x^2)f(x) = 0.$$

Consequently, we see from (7.6), that

$$\lim_{x \rightarrow \pm 1} (1-x^2)^2 f''(x) = 0$$

and, hence

$$(7.7) \quad \lim_{x \rightarrow \pm 1} (1-x)^2 f''(x) = 0.$$

We are now in position to prove part (d). We show that

$$(7.8) \quad \lim_{x \rightarrow 1} (1-x^2) f'(x) = 0;$$

a similar argument establishes the limit as  $x \rightarrow -1$ . Let  $\varepsilon > 0$ . From (7.7), there exists  $x^* \in (0, 1)$  such that

$$|(1-x)^2 f''(x)| < \frac{\varepsilon}{2} \text{ for } x \in [x^*, 1].$$

Integrating this inequality over  $[x^*, x] \subset [x^*, 1]$  yields

$$\frac{\varepsilon}{2(1-x^*)} + f'(x^*) - \frac{\varepsilon}{2(1-x)} < f'(x) < \frac{\varepsilon}{2(1-x)} + f'(x^*) - \frac{\varepsilon}{2(1-x^*)} \text{ for } x \in [x^*, 1].$$

Multiplying this inequality by  $(1-x^2)$  yields

$$(7.9) \quad (1-x^2) \left( f'(x^*) + \frac{\varepsilon}{2(1-x^*)} \right) - \frac{\varepsilon(1+x)}{2} < (1-x^2) f'(x) < \frac{\varepsilon(1+x)}{2} + (1-x^2) \left( f'(x^*) - \frac{\varepsilon}{2(1-x^*)} \right).$$

Letting  $x \rightarrow 1$ , we obtain

$$-\varepsilon \leq \lim_{x \rightarrow 1} (1-x^2) f'(x) \leq \varepsilon$$

and this establishes (7.8). This completes the proof that  $\mathcal{D}(S) \subset \mathcal{D}(A^2)$ .

$\mathcal{D}(A^2) \subset \mathcal{D}(S)$ :

Let  $f \in \mathcal{D}(A^2)$ . Then  $f \in \mathcal{D}(A)$  so

$$(7.10) \quad f, f' \in AC_{\text{loc}}(-1, 1)$$

and

$$(7.11) \quad f \in L^2(-1, 1).$$

Moreover, since  $\ell[f] \in \mathcal{D}(A)$ , it follows that

$$(7.12) \quad \ell^2[f] = \ell[\ell[f]] \in L^2(-1, 1),$$

$$(7.13) \quad \ell[f] = -(1-x^2)f'' + 2xf' \in AC_{\text{loc}}(-1, 1)$$

and

$$(7.14) \quad \ell'[f] = -(1-x^2)f''' + 4xf' + 2f' \in AC_{\text{loc}}(-1, 1).$$

It is clear that if  $f, g \in AC_{\text{loc}}(-1, 1)$  then

- (a)'  $f + g \in AC_{\text{loc}}(-1, 1)$ ;
- (b)'  $fg \in AC_{\text{loc}}(-1, 1)$ ;
- (c)' If  $g > 0$  on  $(-1, 1)$  then  $f/g \in AC_{\text{loc}}(-1, 1)$ .

In particular, from (7.10) and (b)', we see that  $2xf' \in AC_{\text{loc}}(-1, 1)$ . Combining this with (a)' and (7.13), we obtain  $(1-x^2)f'' \in AC_{\text{loc}}(-1, 1)$ . Since  $1-x^2 > 0$  on  $(-1, 1)$  we infer from (c)' that

$$(7.15) \quad f'' \in AC_{\text{loc}}(-1, 1).$$

Continuing,  $-4xf'' - 2f' \in AC_{\text{loc}}(-1, 1)$  so from (a)' and (7.14), we have  $(1-x^2)f''' \in AC_{\text{loc}}(-1, 1)$  and it then follows that

$$(7.16) \quad f''' \in AC_{\text{loc}}(-1, 1).$$

By definition of  $\mathcal{D}(A)$  and the fact that  $\ell[f] \in \mathcal{D}(A)$ , we see that

$$\lim_{x \rightarrow \pm 1} (1 - x^2) \ell'[f](x) = 0;$$

consequently, in view of (7.1), we see that

$$(7.17) \quad 0 = \lim_{x \rightarrow \pm 1} [f, 1]_2(x) = \lim_{x \rightarrow \pm 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2) f'(x) \right).$$

Furthermore since  $f \in \mathcal{D}(A)$ , we have

$$(7.18) \quad \lim_{x \rightarrow \pm 1} (1 - x^2) f'(x) = 0$$

so, from (7.17), we see that

$$(7.19) \quad \lim_{x \rightarrow \pm 1} ((1 - x^2)^2 f''(x))'' = 0.$$

To finish the proof, we need to show that

$$(7.20) \quad \begin{aligned} 0 &= [f, x]_2(\pm 1) = \lim_{x \rightarrow \pm 1} (x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x)) \\ &= \lim_{x \rightarrow \pm 1} (-(1 - x^2)^2 f''(x) + 2(1 - x^2) f(x)) \text{ by (7.17).} \end{aligned}$$

We note again, from Green's formula (4.3), that the limits in (7.20) exist and are finite. Since  $f \in \mathcal{D}(A)$ , we see from Theorem 1, part (v) that  $f \in AC[-1, 1]$  and hence

$$(7.21) \quad \lim_{x \rightarrow \pm 1} (1 - x^2) f(x) = 0.$$

Thus, proving (7.20) reduces to showing

$$(7.22) \quad \lim_{x \rightarrow \pm 1} (1 - x^2)^2 f''(x) = 0.$$

We show that

$$(7.23) \quad \lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) = 0;$$

a similar argument will show

$$\lim_{x \rightarrow -1} (1 - x^2)^2 f''(x) = 0.$$

Suppose, to the contrary, that

$$\lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) = c \neq 0;$$

without loss of any generality, we can suppose that  $c > 0$ . Then there exists  $x^* \in (0, 1)$  such that

$$(1 - x^2)^2 f''(x) \geq r := \frac{c}{2} \text{ on } [x^*, 1];$$

that is,

$$f''(x) \geq \frac{R}{(1 - x)^2} \text{ on } [x^*, 1]$$

for some  $R > 0$ . Integrating this inequality over  $[x^*, x] \subset [x^*, 1]$  yields

$$\begin{aligned} f'(x) &\geq R \int_{x^*}^x \frac{dt}{(1 - t)^2} + f'(x^*) \\ &= \frac{R}{1 - x} + f'(x^*) - \frac{R}{1 - x^*}. \end{aligned}$$

Consequently,

$$\begin{aligned} (1-x^2)f'(x) &\geq R(1+x) + (1-x^2) \left( f'(x^*) - \frac{R}{1-x^*} \right) \\ &\rightarrow 2R > 0 \quad (\text{as } x \rightarrow 1) \end{aligned}$$

contradicting (7.18). It follows that (7.23) holds and this proves (7.20). Combining (7.10), (7.11), (7.12), (7.15), (7.16), (7.17) and (7.20), we see that  $f \in \mathcal{D}(A^2)$  implies  $f \in \mathcal{D}(S)$ . This completes the proof of the theorem.  $\square$

## 8. PROOF OF THEOREM 4

In order to prove Theorem 4, we first need to establish three preliminary facts, the first of which is the following result.

**Lemma 1.** *If  $f \in \mathcal{D}(S)$ , then*

$$(8.1) \quad \frac{1}{1-x^2} \left( (1-x^2)^2 f''(x) \right)' \in L^2(-1, 1).$$

*Proof.* Let  $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$  so  $f' \in L^2(-1, 1)$ ,  $[f, 1]_2(\pm 1) = 0$  and  $\ell^2[f] \in L^2(-1, 1)$ . We apply the CE Theorem on  $[0, 1)$  with  $\psi(x) = 1$ ,  $\varphi(x) = -1/(1-x^2)$  and  $w(x) = 1$ . These functions satisfy the conditions of this theorem on  $[0, 1)$  so

$$\frac{-1}{1-x^2} \int_x^1 \ell^2[f](t) dt \in L^2(0, 1).$$

However, using (4.9), a calculation shows

$$\begin{aligned} \frac{-1}{1-x^2} \int_x^1 \ell^2[f](t) dt &= \frac{-1}{1-x^2} \int_x^1 \left[ \left( (1-t^2)^2 f''(t) \right)' - 2 \left( (1-t^2) f'(t) \right)' \right] dt \\ &= \frac{-1}{1-x^2} \left[ \lim_{x \rightarrow 1} \left( \left( (1-x^2)^2 f''(x) \right)' - 2(1-x^2) f'(x) \right) \right] \\ &\quad + \frac{1}{1-x^2} \left[ \left( (1-x^2)^2 f''(x) \right)' - 2(1-x^2) f'(x) \right] \\ &= \frac{-1}{1-x^2} \left[ \lim_{x \rightarrow 1} [f, 1]_2(x) - \left( (1-x^2)^2 f''(x) \right)' + 2(1-x^2) f'(x) \right] \\ &= \frac{1}{1-x^2} \left( (1-x^2)^2 f''(x) \right)' - 2f'(x). \end{aligned}$$

A similar calculation shows that

$$\frac{1}{1-x^2} \left( (1-x^2)^2 f''(x) \right)' - 2f'(x) \in L^2(-1, 0]$$

and hence

$$\frac{1}{1-x^2} \left( (1-x^2)^2 f''(x) \right)' - 2f'(x) \in L^2(-1, 1).$$

Since  $f' \in L^2(-1, 1)$ , we see, by linearity, that

$$\frac{1}{1-x^2} \left( (1-x^2)^2 f''(x) \right)' \in L^2(-1, 1).$$

$\square$

**Lemma 2.** *For  $f \in \mathcal{D}(S)$ , we have*

$$(8.2) \quad \lim_{x \rightarrow \pm 1} (1-x^2)^2 f''(x) = 0.$$

*Proof.* Let  $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$ . Since  $f \in \mathcal{D}(A)$ , we have  $f \in AC[-1, 1]$  so

$$(8.3) \quad \lim_{x \rightarrow \pm 1} (1 - x^2)f(x) = 0.$$

Furthermore, we have

$$(8.4) \quad 0 = \lim_{x \rightarrow \pm 1} [f, 1]_2(x) = \lim_{x \rightarrow \pm 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \right).$$

Consequently, from (8.3) and (8.4), we find that

$$0 = \lim_{x \rightarrow \pm 1} [f, x]_2(x) = \lim_{x \rightarrow \pm 1} (x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2)f(x)) = - \lim_{x \rightarrow \pm 1} (1 - x^2)^2 f''(x).$$

□

The last preliminary result is the following theorem. Since  $\mathcal{D}(S) = \mathcal{D}(A^2)$ , this next result generalizes the well-known result for  $\mathcal{D}(A)$  established in Theorem 1, part (v).

**Theorem 8.** *If  $f \in \mathcal{D}(S)$ , then*

$$f'' \in L^2(-1, 1).$$

*Moreover,*

$$(8.5) \quad pf'' \in L^2(-1, 1)$$

*for any bounded, Lebesgue measurable function  $p$ , including any polynomial.*

*Proof.* Once we establish  $f'' \in L^2(-1, 1)$ , the statement in (8.5), for any bounded measurable function, follows clearly. Let  $f \in \mathcal{D}(S)$ . We prove that  $f'' \in L^2(0, 1)$ ; a similar proof will establish  $f'' \in L^2(-1, 0)$  and prove the theorem. We again use the CE Theorem with  $\psi(x) = 1 - x^2$ ,  $\varphi(x) = 1/(1 - x^2)^2$  and  $w(x) = 1$  on  $[0, 1)$ . Indeed, from the CE Theorem and (8.1), we find that

$$\frac{-1}{(1 - x^2)^2} \int_x^1 (1 - t^2) \left( \frac{1}{1 - t^2} ((1 - t^2)^2 f''(t))' \right) dt \in L^2(0, 1).$$

However, from Lemma 2,

$$\begin{aligned} & \frac{-1}{(1 - x^2)^2} \int_x^1 (1 - t^2) \left( \frac{1}{1 - t^2} ((1 - t^2)^2 f''(t))' \right) dt \\ &= \frac{-1}{(1 - x^2)^2} \left( \lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) - (1 - x^2)^2 f''(x) \right) \\ &= f''(x). \end{aligned}$$

□

We are now in position to prove Theorem 4, specifically  $B = \mathcal{D}(S)$ , where  $B$  is defined in (1.2) and  $\mathcal{D}(S)$  is given in (1.5).

*Proof.*  $B \subset \mathcal{D}(S)$ :

Let  $f \in B$ . We assume that  $f$  is real-valued on  $(-1, 1)$ . We begin by showing, using the CE Theorem, that the condition

$$(1 - x^2)^2 f^{(4)} \in L^2(-1, 1)$$

implies the two conditions

$$(8.6) \quad (1 - x^2)f''' \in L^2(-1, 1)$$

and

$$(8.7) \quad f'' \in L^2(-1, 1).$$

Regarding (8.6), we will show

$$(8.8) \quad (1 - x^2)f''' \in L^2(0, 1);$$

a similar proof will yield

$$(8.9) \quad (1 - x^2)f''' \in L^2(-1, 0)$$

and, together, they establish (8.6). Since  $(1 - x^2)^2 f^{(4)} \in L^2(0, 1)$ , we use the CE Theorem on  $[0, 1)$  with

$$\varphi(x) = (1 - x^2)^{-2}, \quad \psi(x) = 1 - x^2 \text{ and } w(x) = 1 \quad (x \in [0, 1)).$$

It follows that

$$\begin{aligned} (1 - x^2)f'''(x) &= (1 - x^2) \int_0^x \frac{1}{(1 - t^2)^2} (1 - t^2)^2 f^{(4)}(t) dt + f'''(0)(1 - x^2) \\ &\in L^2(0, 1). \end{aligned}$$

To see (8.7), we apply the CE Theorem once again on  $[0, 1)$  to prove that

$$f'' \in L^2(0, 1);$$

a similar argument will show that  $f'' \in L^2(-1, 0)$ . To this end, let

$$\varphi(x) = (1 - x^2)^{-1}, \quad \psi(x) = 1 \text{ and } w(x) = 1 \quad (x \in [0, 1)).$$

In this case, we see that

$$f''(x) = \int_0^x \frac{1}{1 - t^2} ((1 - t^2)f'''(t)) dt + f''(0) \in L^2(0, 1).$$

Consequently, we see that

$$f, f' \in AC[-1, 1] \subset L^2(-1, 1).$$

Moreover, it is clear that  $g(x)(1 - x^2)f'''(x)$ ,  $g(x)f''(x)$  and  $g(x)f'(x)$  all belong to  $L^2(-1, 1)$  for any bounded, measurable function  $g$  on  $(-1, 1)$ . Hence

$$\begin{aligned} \ell^2[f](x) &= (1 - x^2)^2 f^{(4)}(x) - 8x(1 - x^2)f'''(x) + (14x^2 - 6)f''(x) + 4xf'(x) \\ &\in L^2(-1, 1) \end{aligned}$$

and, in particular,

$$(8.10) \quad 4xf' \in L^2(-1, 1).$$

It remains to show that

$$(8.11) \quad \lim_{x \rightarrow \pm 1} [f, 1]_2(x) = \lim_{x \rightarrow \pm 1} [f, x]_2(x) = 0.$$

Since  $1, x \in \Delta_{2, \max}$ , we see from Green's formula in (4.3) that the limits in (8.11) both exist and are finite. Now  $f' \in AC[-1, 1]$  so

$$\lim_{x \rightarrow \pm 1} (1 - x^2)f'(x) = 0.$$

Consequently,

$$\begin{aligned} \lim_{x \rightarrow \pm 1} [f, 1]_2(x) &= \lim_{x \rightarrow \pm 1} (((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x)) \\ &= \lim_{x \rightarrow \pm 1} ((1 - x^2)^2 f''(x))'. \end{aligned}$$

We claim that

$$(8.12) \quad \lim_{x \rightarrow 1} ((1 - x^2)^2 f''(x))' = 0;$$

a similar proof will establish

$$\lim_{x \rightarrow -1} ((1 - x^2)^2 f''(x))' = 0.$$

Suppose to the contrary that

$$\lim_{x \rightarrow 1} ((1 - x^2)^2 f''(x))' = c \neq 0;$$

we can assume that  $c > 0$ . It follows that there exists  $x^* \in (0, 1)$  such that

$$(8.13) \quad ((1 - x^2)^2 f''(x))' \geq r := \frac{c}{2} > 0 \quad (x \in [x^*, 1)).$$

Note that since

$$((1 - x^2)^2 f''(x))' = (1 - x^2)^2 f'''(x) - 4x(1 - x^2) f''(x),$$

we see that the inequality in (8.13) can be rewritten as

$$(8.14) \quad (1 - x^2) f'''(x) - 4x f''(x) \geq \frac{r}{1 - x^2} \text{ on } [x^*, 1).$$

However, from (8.5) and (8.6), we see that

$$(1 - x^2) f'''(x) - 4x f''(x) \in L^2(-1, 1)$$

so the inequality in (8.14) is not possible. Hence (8.12) is established and thus

$$\lim_{x \rightarrow \pm 1} [f, 1]_2(x) = 0.$$

We now show that

$$\lim_{x \rightarrow \pm 1} [f, x]_2(x) = 0.$$

Since the argument for  $x \rightarrow -1$  mirrors the proof for  $x \rightarrow 1$ , we will only show that

$$(8.15) \quad \lim_{x \rightarrow 1} [f, x]_2(x) = 0.$$

Now, since  $f \in AC[-1, 1]$ , we see that  $\lim_{x \rightarrow 1} (1 - x^2) f(x) = 0$ ; moreover, using (8.12),

$$\lim_{x \rightarrow 1} [f, x]_2(x) = \lim_{x \rightarrow 1} (x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x)) = - \lim_{x \rightarrow 1} (1 - x^2)^2 f''(x).$$

Suppose that

$$\lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) = d \neq 0;$$

we can assume that  $d > 0$ . Then, with possibly different  $x^*$  as given in the above argument, there exists a  $x^* \in (0, 1)$  with

$$(1 - x^2)^2 f''(x) \geq d' := \frac{d}{2} \quad (x \in [x^*, 1)).$$

Hence

$$f''(x) \geq \frac{d'}{(1 - x^2)^2} \quad (x \in [x^*, 1)).$$

However, this implies that  $f'' \notin L^2(0, 1)$ , contradicting (8.7). Thus (8.15) is established and this completes the proof that  $B \subset \mathcal{D}(S)$ .

$\mathcal{D}(S) \subset B$ :

Let  $f \in \mathcal{D}(S)$ . We need only to show that

$$(8.16) \quad (1 - x^2)^2 f^{(4)} \in L^2(-1, 1).$$

Since, by Theorem 8,  $f'' \in L^2(-1, 1)$ , we see that  $g f'' \in L^2(-1, 1)$  for any bounded, measurable function  $g$  on  $(-1, 1)$ . In particular, it is the case that

$$(8.17) \quad 4x f'' \in L^2(-1, 1)$$



and

$$(8.18) \quad (14x^2 - 6)f'' \in L^2(-1, 1).$$

By (8.1),

$$(8.19) \quad (1 - x^2)f'''(x) - 4xf''(x) = \frac{1}{1 - x^2} ((1 - x^2)^2 f''(x))' \in L^2(-1, 1).$$

By linearity, it follows from (8.17) and (8.19) that

$$(1 - x^2)f''' \in L^2(-1, 1).$$

Consequently,  $g(1 - x^2)f''' \in L^2(-1, 1)$  for every bounded, measurable function  $g$  on  $(-1, 1)$ ; in particular,

$$(8.20) \quad 8x(1 - x^2)f''' \in L^2(-1, 1).$$

Furthermore, since  $f' \in L^2(-1, 1)$ , it follows that

$$(8.21) \quad 4xf'(x) \in L^2(-1, 1).$$

Finally, since  $\ell^2[f] \in L^2(-1, 1)$ , we see from (3.2), (8.18), (8.20) and (8.21) that

$$\begin{aligned} (1 - x^2)^2 f^{(4)} &= \ell^2[f] + 8x(1 - x^2)f''' - (14x^2 - 6)f'' - 4xf' \\ &\in L^2(-1, 1). \end{aligned}$$

This establishes (8.16) and proves  $\mathcal{D}(S) \subset B$ . This completes the proof of Theorem 4.  $\square$

## 9. PROOF OF THEOREM 5

We now prove Theorem 5, namely  $\mathcal{D}(S) = D$ , where  $\mathcal{D}(S)$  is given in (1.5) and  $D$  is defined in (1.6).

*Proof.* Since functions  $f$  in both  $\mathcal{D}(S)$  and  $D$  satisfy the ‘maximal domain’ conditions  $f^{(j)} \in AC_{\text{loc}}(-1, 1)$  ( $j = 0, 1, 2, 3$ ),  $f \in L^2(-1, 1)$  and  $\ell^2[f] \in L^2(-1, 1)$ , we need only to prove that the other properties in their definitions hold.

$\mathcal{D}(S) \subset D$ :

Let  $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$ . Then  $f \in \mathcal{D}(A)$  so

$$(9.1) \quad \lim_{x \rightarrow \pm 1} (1 - x^2)f'(x) = 0.$$

Moreover,

$$\begin{aligned} 0 &= [f, 1]_2(\pm 1) \\ &= \lim_{x \rightarrow \pm 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \right) \\ (9.2) \quad &= \lim_{x \rightarrow \pm 1} ((1 - x^2)^2 f''(x))'. \end{aligned}$$

The identities in (9.1) and (9.2) prove that  $\mathcal{D}(S) \subset D$ .

$D \subset \mathcal{D}(S)$ :

Let  $f \in D$ . Clearly,

$$\begin{aligned} (9.3) \quad [f, 1]_2(\pm 1) &= \lim_{x \rightarrow \pm 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \right) \\ &= 0 \end{aligned}$$

so we need to show that

$$(9.4) \quad \lim_{x \rightarrow \pm 1} [f, x]_2(\pm 1) = 0.$$

We remark that the limits in (9.4) exist (by Green's formula) and are finite.

Claim:  $\ell'[f] \in L^2(-1, 1)$ .

To see this, recall the two representations of  $\ell^2[\cdot]$ : the one given in (3.2) and the one given in (7.2). Since  $\ell^2[f] \in L^2(-1, 1)$ , we apply the CE Theorem on  $[0, 1]$  with  $\varphi(x) = (1 - x^2)^{-1}$ ,  $\psi(x) = 1$  and  $w(x) = 1$  to obtain

$$\frac{1}{1 - x^2} \int_x^1 \ell^2[f](t) dt \in L^2(0, 1).$$

However, from (3.2) and (7.2), we see that

$$\begin{aligned} & \frac{1}{1 - x^2} \int_x^1 \ell^2[f](t) dt \\ &= \frac{1}{1 - x^2} \left( \lim_{x \rightarrow 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2) f'(x) \right) + (1 - x^2) \ell'[f](x) \right) \\ &= \ell'[f](x) \text{ by (9.3);} \end{aligned}$$

a similar calculation shows that  $\ell'[f] \in L^2(-1, 0)$ . It follows that  $\ell[f] \in AC[-1, 1] \subset L^2(-1, 1)$ . We again apply the CE Theorem on  $[0, 1]$  with  $\varphi(x) = (1 - x^2)^{-1}$ ,  $\psi(x) = 1$  and  $w(x) = 1$  to obtain

$$\frac{1}{1 - x^2} \int_x^1 \ell[f](t) dt \in L^2(0, 1).$$

Another calculation shows that

$$\begin{aligned} & \frac{1}{1 - x^2} \int_x^1 \ell[f](t) dt = \frac{-1}{1 - x^2} \int_x^1 ((1 - t^2) f'(t))' dt \\ &= \frac{-1}{1 - x^2} \left( \lim_{x \rightarrow 1} (1 - x^2) f'(x) - (1 - x^2) f'(x) \right) \\ &= f'(x) \text{ by definition of } D; \end{aligned}$$

a similar argument shows that  $f' \in L^2(-1, 0)$ . Hence

$$(9.5) \quad f' \in L^2(-1, 1).$$

Thus,  $f \in AC[-1, 1]$  and

$$(9.6) \quad \lim_{x \rightarrow \pm 1} (1 - x^2) f(x) = 0.$$

From (9.3) and (9.6), we see that

$$\begin{aligned} \lim_{x \rightarrow \pm 1} [f, x]_2(x) &= \lim_{x \rightarrow \pm 1} (x[f, 1](x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x)) \\ &= - \lim_{x \rightarrow \pm 1} (1 - x^2)^2 f''(x). \end{aligned}$$

To establish (9.4), it now suffices to prove that

$$(9.7) \quad \lim_{x \rightarrow \pm 1} (1 - x^2)^2 f''(x) = 0.$$

Since the proof as  $x \rightarrow -1$  is similar to the proof that  $x \rightarrow 1$ , we will only show that

$$\lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) = 0;$$

By way of contradiction, suppose that

$$\lim_{x \rightarrow 1} (1 - x^2)^2 f''(x) = c \neq 0;$$

without loss of generality, we may assume that  $c > 0$ . Then there exists  $x^* \in (0, 1)$  such that

$$f''(x) \geq \frac{c}{2(1-x^2)^2} \geq \frac{c}{8(1-x)^2} \quad (x \in [x^*, 1)).$$

Integrating this inequality over  $[x^*, x] \subset [x^*, 1)$  yields

$$f'(x) \geq \frac{c}{8(1-x)} + f'(x^*) - \frac{c}{8(1-x^*)} \quad (x \in [x^*, 1)).$$

But this contradicts (9.5). It follows that (9.7) holds and this, in turn, establishes (9.4). Consequently,  $D \subset \mathcal{D}(S)$  and this completes the proof of the theorem.  $\square$

As revealed in the proofs of Theorems 3, 4, 5 and 8, we have the following interesting result.

**Corollary 1.** *If  $f \in \mathcal{D}(A^2) = \mathcal{D}(S) = B = D$ , then*

- (i)  $f'' \in L^2(-1, 1)$  so  $f, f' \in AC[-1, 1]$ ;
- (ii)  $\ell'[f] \in L^2(-1, 1)$  and  $\ell[f] \in AC[-1, 1]$ .

**Remark 2.** *As discussed in Section 4, the minimal operator  $T_{2,\min}$  in  $L^2(-1, 1)$  generated by  $\ell^2[\cdot]$  has deficiency index  $(4, 4)$ . From the GKN Theorem (see [17, Theorem 4, Section 18.1]), GKN boundary conditions for any self-adjoint extension of  $T_{2,\min}$  in  $L^2(-1, 1)$  are restrictions of the maximal domain  $\Delta_{2,\max}$  and have the appearance (see (4.5))*

$$[f, f_j]_2(1) - [f, f_j]_2(-1) = 0 \quad (f \in \Delta_{2,\max}, j = 1, 2, 3, 4),$$

where  $\{f_j\}_{j=1}^4 \subset \Delta_{2,\max}$  are linearly independent modulo the minimal domain  $\Delta_{2,\min}$ . Taking into account  $[\cdot, \cdot]_2$ , defined in (4.2), it is clear that the boundary conditions given in (1.6) are not GKN boundary conditions.

## 10. CONCLUDING REMARKS

In [11], the authors showed that, for  $n \in \mathbb{N}$ , the  $n^{\text{th}}$  composite power of the Legendre differential expression  $\ell[\cdot]$  is explicitly given by

$$(10.1) \quad \ell^n[y](x) = \sum_{j=1}^n (-1)^j \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_1 \left( (1-x^2)^j y^{(j)}(x) \right)^{(j)},$$

where the numbers

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\}_1 := \sum_{r=0}^j (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(j-r)!(j+r+1)!}$$

are the so-called Legendre-Stirling numbers, a subject of current study in combinatorics (for example, see [2], [3], [4], [7] and [12]). The expression in (10.1) is the key in generating the domain  $\mathcal{D}(A^n)$  of  $A^n$  given in (1.1).

We conjecture:

Conjecture Let  $A$  denote the Legendre polynomials self-adjoint operator defined in (2.4) and (2.5). For  $n \in \mathbb{N}$ , let  $\ell^n[\cdot]$  be given as in (10.1) and let  $[\cdot, \cdot]_n$  be the sesquilinear form associated with the maximal domain  $\Delta_{n,\max}$  of  $\ell^n[\cdot]$  in  $L^2(-1, 1)$ . Then  $A_n = B_n = C_n = D_n$ , where

- (i)  $A_n := \mathcal{D}(A^n)$ ,
- (ii)  $B_n := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', \dots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); (1-x^2)^n f^{(2n)} \in L^2(-1, 1)\}$ ,
- (iii)  $C_n := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', \dots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); f, \ell^n[f] \in L^2(-1, 1);$   
 $[f, x^j]_n(\pm 1) = 0 \text{ for } j = 0, 1, 2, \dots, n-1\}$ ,
- (iv)  $D_n := \{f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', \dots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); f, \ell^n[f] \in L^2(-1, 1);$   
 $\lim_{x \rightarrow \pm 1} ((1-x^2)^j y^{(j)}(x))^{(j-1)} = 0 \text{ for } j = 1, 2, \dots, n\}$ .

By repeated applications of the CE Theorem, it is not difficult to establish that if  $f \in B_n$ , then  $f^{(n)} \in L^2(-1, 1)$ ; this result generalizes Theorem 1, part (iii) ( $n = 1$ ) and Corollary 1, part (i) ( $n = 2$ ).

We remark that, in (iii) above, we can replace the monomials  $\{x^j\}_{j=0}^{n-1}$  by the Legendre polynomials  $\{P_j\}_{j=0}^{n-1}$ . One of the difficulties in our efforts to try and prove this conjecture lies in the fact that the corresponding sesquilinear form  $[\cdot, \cdot]_n$ , associated with the  $n^{\text{th}}$  power  $\ell^n[\cdot]$ , is unwieldy at the present time.

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